Lecture Notes II-2 Dynamic Games of Complete Information

- Extensive Form Representation (Game tree)
- Subgame Perfect Nash Equilibrium
- Perfect Information and Imperfect Information
- Repeated Games
- Trigger Strategy

Dynamic Games of Complete Information

- Dynamic game with complete information
  - Sequential games in which the players' payoff functions are common knowledge
  - Perfect (imperfect) information: For each move in the play of the game, the player with the move knows (doesn't know) the full history of the play of the game so far

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Dynamic Game of Complete and Perfect Information

- Key features
  - (1) the moves occur in sequence
  - (2) all previous moves are observed before the next move is chosen
  - (3) the players' playoff from each feasible combination of moves are common knowledge

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Backwards Induction

- A simple dynamic game of complete and perfect information
  - 1. Player 1 chooses an action a_1 from the feasible set A_1
  - 2. Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2
  - Payoffs are $u_1(a_1,a_2)$ and $u_2(a_1,a_2)$

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Backwards Induction (cont')

- The player 2's optimization problem in the second stage

$$\max_{a_2 \in A_2} u_2(a_1,a_2)$$

  - Assume that for each a_1 in A_1, players' optimization problem has a unique solution, denoted by $R_2(a_1)$. This player 2's reaction (or best response) to player 1's action

- The player 1's optimization problem in the first stage

$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$$

  - Assume that this optimization problem for player 1 also has a unique solution denoted by a_1^*
  - We call $(a_1^*, R_2^*(a_1^*))$ the backwards induction outcome of this game

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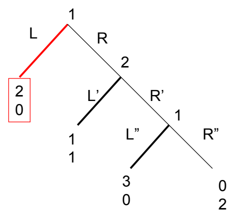
Extensive-Form Representation



In the first stage, player 1 play the optimal action *L*

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Example 1: Stackelberg Model of Duopoly

- Timing of the game
  - (1) firm 1 chooses a quantity q_1
  - (2) firm 2 observes q_1 then choose a quantity q_2
- Demand function
  - $P(Q)=a-Q$, $Q=q_1+q_2$
- Profit function to firm i
  - $\pi(q_i,q_j)=q_i[P(Q)-c]$

Example 1: Stackelberg Model of Duopoly (cont')

- In the second stage, firm 2's reaction to an arbitrary quantity by firm 1 $R_2(q_1)$ is given by solving

$$\max_{q_2\ge 0}\pi_2(q_1,q_2)=\max_{q_2\ge 0} q_2[a-q_1-q_2-c]$$

$$\Rightarrow R_2(q_1)=\frac{a-q_1-c}{2}$$

- In the first stage, firm 1's problem is to solve

$$\max_{q_1\ge 0}\pi_1(q_1,R_2(q_1))=\max_{q_1\ge 0} q_1[a-q_1-R_2(q_1)-c]=\max_{q_1\ge 0} q_1\left(\frac{a-q_1-c}{2}\right)$$

$$\Rightarrow q_1^*=\frac{a-c}{2}$$

- Outcome $q_1^*=\frac{a-c}{2}$ and $R_2(q_1^*)=\frac{a-c}{4}$

Example 1: Stackelberg Model of Duopoly (cont')

- Compare with Nash equilibrium of the simultaneous Cournot game

Decide simultaneously

$$q_1^*=q_2^*=\frac{a-c}{3}, Q^*=\frac{2(a-c)}{3}, p=\frac{a+2c}{3}$$

Decide sequentially

$$q_1^*=\frac{a-c}{2}, R_2(q_1^*)=\frac{a-c}{4}, Q^*=\frac{3(a-c)}{4}, p=\frac{a+3c}{4}$$

Example 1: Stackelberg Model of Duopoly (cont')

Decide simultaneously

$$\pi_1^*=\pi_2^*=\frac{a-c}{3}\cdot\frac{a-c}{3}=\frac{(a-c)^2}{9}$$

Decide sequentially

$$\pi_1^*=\frac{a-c}{2}\cdot\frac{a-c}{4}=\frac{(a-c)^2}{8}, \pi_2^*=\frac{a-c}{4}\cdot\frac{a-c}{4}=\frac{(a-c)^2}{16}$$

⇨ In single-person decision theory, having more information can never make the decision worse off, In game theory, however, having more information can make a player worse off

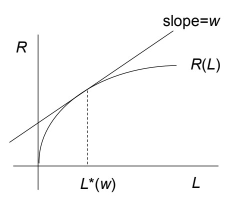
Example 2: Wages and Employment in a Unionized Firm

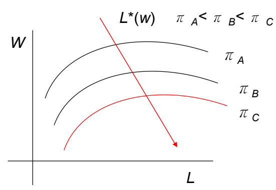
- Timing of the game
  - (1) the union makes a wage demand, w;
  - (2) the firm observes (and accept) w and then chooses employment L;
  - (3) union and firm's payoffs are $U(w,L)$ and $\pi(w,L)=R(L)-wL$ respectively
    - $U(w,L)$ increases with w and L
    - $R(L)$ is increasing and concave

Example 2: Wages and Employment in a Unionized Firm (cont')

In the second stage, the firm chooses $L^*(w)$ to solve

$$\max_{L\ge 0}\pi(w,L)=\max_{L\ge 0} R(L)-wL \quad \Rightarrow \quad R'(L)-w=0$$



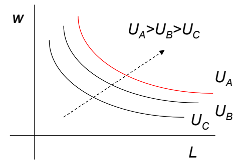


Example 2: Wages and Employment in a Unionized Firm (cont')

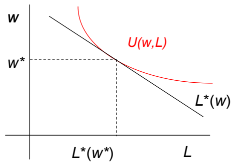
In the first stage, the union chooses w^* to solve

$$\max_{w\ge 0} U(w, L^*(w))$$

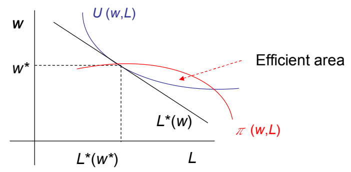
Union indifference curve



Union indifference curve



Example 2: Wages and Employment in a Unionized Firm (cont')



⇨ (w^*, $L^*(w^*)$) is inefficient

Example 3: Sequential Bargaining

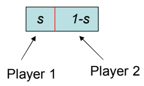
Stage 1: player 1 offers (s_1,1-s_1)

Player 2 accept (s_1,1-s_1)

Stage 2: player 2 offer (s_2,1-s_2)

Player 1 accept (s_2,1-s_2)

Stage 3: allocation
(s, (1-s))



Example 3: Sequential Bargaining (cont')

Stage 1: player 1 offers (s_1,1-s_1)

Player 2 accepts 1-s_1 if $1-s_1^* \geq \delta(1-s_2^*)$

Stage 2: player 2 offer (s_2,1-s_2)

⇨ (δs_2^*, $\delta(1-s_2^*)$)

Player 1 accept s_2 if $s_2^* \geq \delta s$

Stage 3: allocation

(s, (1-s)) ⇨ (δs, δ(1-s))

Outcome: player 1 offers $s_1^*= 1-\delta(1-\delta s)$ and player 2 accepts

Example 3: Sequential Bargaining (cont')

Stage 1: player 1 offers (s_1,1-s_1)

(s_1,1-s_1) Player 2 accept

Stage 2: player 2 offer (s_2,1-s_2)

(s_2,1-s_2) Player 1 accept

Stage 3: player 1 offers (s_3,1-s_3)

⇨ $s_1^*= 1-\delta(1-\delta s_1)=1-\delta+\delta^2 s_1$

infinite ⇨ $s_1^*=1/(1+\delta)$, $1-s^*=\delta/(1+\delta)$

Two-Stage Game of Complete but Imperfect Information

- A two-stage game
  - Players 1 and 2 simultaneously choose actions a_1 and a_2 from feasible sets A_1 and A_2, respectively
  - Players 3 and 4 observe the outcome of the first stage, (a_1,a_2), and then simultaneously choose actions a_3 and a_4 from feasible sets A_3 and A_4, respectively
  - Payoffs are $u_i(a_1,a_2,a_3,a_4)$ for i=1,2,3,4

Two-Stage Game of Complete but Imperfect Information (cont')

- Backward induction
  - For any feasible outcome of the first-stage game, (a_1,a_2), the second–stage that remains between players 3 and 4 has a unique Nash equilibrium $(a_3^*(a_1,a_2),\ a_4^*(a_1,a_2))$
- Subgame-perfect outcome
  - Suppose (a_1^*,a_2^*) is the unique Nash equilibrium of simultaneous-move game of player 1 and player 2
  - $(a_1^*,a_2^*,a_3^*(a_1^*,a_2^*),a_4^*(a_1^*,a_2^*))$ is called subgame-perfect outcome

Example 1: Bank Runs

- Two investors has each deposited D with a bank
- The bank has invested these deposits in a long-term project.
  - If the bank is forced to liquidate its investment before long-term matures, a total $2r$ will be recovered
  - If the bank allows the investment to reach maturity, the project will pay out a total $2R$
  - The investors can make withdraw from bank at date 1, before the bank's investment mature or date 2, after
- Assumption : $R>D>r>D/2$

Example 1: Bank Runs (cont')

Date 1

	withdraw	don't
withdraw	r,r	$D,2r-D$
don't	$2r-D,\ D$	Next stage

Date 2

	withdraw	don't
withdraw	R,R	$2R-D,D$
don't	$D,2R-D$	R,R

	withdraw	don't
withdraw	r,r	$D,2r-D$
don't	$2r-D,\ D$	R,R

Two subgame-perfect outcomes : (1) both investors withdraw at date 1 (inefficient)
(2) both investors withdraw at date 2 (efficient)

Example 2: Tariffs and Imperfect International Competition

- Two identical countries, denoted by i=1,2
- Each country has a government that chooses a tariff rate t_i, a firm that produces output for both home consumption h_i and export e_i
- If the total quantity on the market in country i is Q_i, then the market-clearing price is $P_i(Q_i)=a-Q_i$, where $Q_i=h_i+e_j$
- The total cost of production for firm i is $C_i(h_i,e_i)=c(h_i+e_i)$

Example 2: Tariffs and Imperfect International Competition (cont')

- Timing of the game
  - First, the governments simultaneously choose tariff rates t_1 and t_2
  - Second, the firms observe the tariff rates and simultaneously choose quantities for home consumption and for export (h_1,e_1) and (h_2,e_2)
- Payoffs are profit to firm i and total welfare to government i
  - welfare =consumers' surplus + firms' profit +tariff revenue

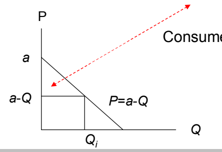
Example 2: Tariffs and Imperfect International Competition (cont')

Firm i's profit

$$\pi_i(t_i,t_j,h_i,e_i,h_j,e_j)=[a-(h_i+e_j)]h_i+[a-(e_i+h_j)]e_i-c(h_i+e_i)-t_je_i$$

Government i's payoff

$$W_i(t_i,t_j,h_i,e_i,h_j,e_j)=\frac{1}{2}Q_i^2+\pi_i(t_i,t_j,h_i,e_i,h_j,e_j)+t_ie_j$$



Example 2: Tariffs and Imperfect International Competition (cont')

Firm i's optimization problem

$$\max_{h_i, e_i} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*)$$

$$\max_{h_i \geq 0} h_i[a-(h_i+e_j^*)-c] \Rightarrow h_j^* = \frac{1}{2}(a-e_j^*-c)$$

$$\max_{e_i \geq 0} e_i[a-(h_j^*+e_i)-c]-t_j e_i \Rightarrow e_i^* = \frac{1}{2}(a-h_j^*-c-t_j)$$

$$\Rightarrow h_i^* = \frac{a-c+t_i}{3} \qquad e_i^* = \frac{a-c-2t_j}{3}$$

Example 2: Tariffs and Imperfect International Competition (cont')

Government i's optimization problem

$$\max_{t_i \geq 0} W_i^*(t_i, t_j^*) = \frac{(2(a-c)-t_i)^2}{18} + \frac{(a-c+t_i)^2}{9} + \frac{(a-c-2t_j^*)^2}{9} + \frac{t_i(a-c-2t_i)}{3}$$

$$\Rightarrow t_i^* = \frac{a-c}{3} \quad h_i^* = \frac{a-c+t_i}{3} = \frac{4(a-c)}{9} \quad e_i^* = \frac{a-c-2t_j}{3} = \frac{a-c}{9}$$

$$Q_i = h_i + e_j = \frac{5(a-c)}{9}$$

Implication

$t_i^* = 0 \Rightarrow Q_i = \frac{2(a-c)}{3}$ (Cournot's model), higher consumers' surplus

$\max_{t_i, t_j \geq 0} W_1^*(t_1, t_2) + W_2^*(t_2, t_2) \Rightarrow t_1^* = t_2^* = 0$ (free trade)

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Example 3: Tournaments

- Two workers and their boss
- Worker i produces output $y_i=e_i+\varepsilon_i$, where e_i is effort and ε_i is noise
- The workers simultaneously choose nonnegative effort levels
- The noise terms ε_1 and ε_2 are independently draws from a density $f(\varepsilon)$ with zero mean
- The workers' outputs are observed but their effort choices are not
- The workers' wages therefore can depend on their outputs but not (directly) on their effort

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Example 3: Tournaments (cont')

- The workers' boss decides to induce effort from the worker by having them compete in tournament
- The wage earned by the winner of the tournament is w_H, the wage earned by the loser is w_L
- The payoff to a worker from earning wage w and expending effort e is $u(w,e)=w-g(e)$, $g(e)$ is a convexly increasing function
- The payoff to the boss is $y_1+y_2-w_H-w_L$

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Example 3: Tournaments (cont')

Worker i's optimization problem

$$\max_{e_i \geq 0} w = w_H \text{Prob}\{y_i(e_i) > y_j(e_j^*)\} + w_L \text{Prob}\{y_i(e_i) \leq y_j(e_j^*)\} - g(e_i)$$

$$= (w_H - w_L)\text{Prob}\{y_i(e_i) > y_j(e_j^*)\} + w_L - g(e_i)$$

First order condition

$$\frac{\partial w}{\partial e_i} = (w_H - w_L)\frac{\text{Prob}\{y_i(e_i) > y_j(e_j^*)\}}{\partial e_i} - g'(e_i)$$

$$= (w_H - w_L)\int_{\varepsilon_j} f(e_j^* - e_i + \varepsilon_j) f(\varepsilon_j) d\varepsilon_j - g'(e_i)$$

$$\text{Prob}\{y_i(e_i) > y_j(e_j^*)\} = \text{Prob}\{\varepsilon_i > e_j^* + \varepsilon_j - e_i\}$$
$$= \int_{\varepsilon_j} \text{Prob}\{\varepsilon_i > e_j^* + \varepsilon_j - e_i \mid \varepsilon_j\} f(\varepsilon_j) d\varepsilon_j$$
$$= \int_{\varepsilon_j} [1 - F(e_j^* - e_i + \varepsilon_j)] f(\varepsilon_j) d\varepsilon_j$$

Symmetric Nash equilibrium $e_1^* = e_2^* = e^*$

$$\Rightarrow (w_H - w_L)\int_{\varepsilon_j} f(\varepsilon_j)^2 d\varepsilon_j = g'(e^*)$$

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Example 3: Tournaments (cont')

Boss's optimization problem

$$\underset{e^*}{Max}\, 2e^* - w_H - w_L \;\; s.t. \frac{1}{2}w_H + \frac{1}{2}w_L - g(e^*) \geq U_a$$

$$\underset{e^*}{Max}\, 2e^* - 2g(e^*) - U_a$$

First order condition $\quad 1 = g'(e^*)$

$\Rightarrow$ Optimal wage w_H^*, w_L^* satisfies $\begin{cases} w_H + w_L - 2g(e^*) = 2U_a \\ (w_H - w_L)\int_{\varepsilon_j} f(\varepsilon_j)^2 d\varepsilon_j = 1 \end{cases}$

Example:

If $f(\varepsilon) \sim N(0, \sigma^2) \Rightarrow \int_{\varepsilon_j} f(\varepsilon_j)^2 d\varepsilon_j = \frac{1}{2\sigma\sqrt{\pi}}$

$\Rightarrow$ *e^* decreases with σ*

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Two-Stage Repeated Game

Player 2

Player 1	L_2	R_2
L_1	1,1	5,0
R_1	0,5	4,4

Prisoner 2

Player 1	L_2	R_2
L_1	2,2	6,1
R_1	1,6	5,5

- The unique subgame-perfect outcome of the two-stage Prisoners' Dilemma is (L_1,L_2) in the first stage, followed by (L_1,L_2) in the second stage
- Cooperation, that is, (R_1,R_2) cannon be achieved in either stage of the subgame-perfect outcome

Finitely Repeated Game

- Definition
  – Given a stage game G, let $G(T)$ denote the finitely repeated game in which G is played T times, with the outcomes of all proceeding plays observed before the next play begins. The playoffs for $G(T)$ are simply the sum of the playoffs from the T stage games
- Proposition
  – If the stage game G has a unique Nash equilibrium then, for any finite T, the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage

Finitely Repeated Game with Multiple Nash Equilibrium

	L_2	M_2	R_2
L_1	1,1	5,0	0,0
M_1	0,5	4,4	0,0
R_1	0,0	0,0	3,3

Two Nash equilibria (L_1,L_2) and (R_1,R_2)

Suppose the players anticipate that (R_1,R_2) will be the second-stage outcome if the first stage outcome is (M_1,M_2), but that (L_1,L_2) will be the second-stage if any of the eight other first stage outcome occurs

	L_2	M_2	R_2
L_1	2,2	6,1	1,1
M_1	1,6	7,7	1,1
R_1	1,1	1,1	4,4

Three subgame perfect Nash outcomes
$((L_1,L_2), (L_1,L_2))$ with payoff (2,2)
$((M_1,M_2), (R_1,R_2))$ with payoff (7,7)
$((R_1,R_2), (L_1,L_2))$ with payoff (4,4)

Finitely Repeated Game with Multiple Nash Equilibrium (cont')

- Cooperation can be achieved in the first stage of a subgame-perfect outcome of the repeated game
- If G is a static game of complete information with multiple Nash equilibria then in which, for any $t<T$, there may be subgame-perfect outcome in stage t is not a Nash equilibrium of G
- Implication: credible threats or promises about future behavior can influence current behavior

Infinitely Repeated Games

- Present value : Given the discount factor δ, the present value of the infinite sequence of payoffs, $\pi_1, \pi_2, \pi_3, \ldots$ is

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t$$

- Trigger strategy: player i cooperates until someone fails to cooperate, which triggers a switch to noncooperation forever after
  – Trigger strategy is Subgame perfect Nash equilibrium when δ is sufficiently large
- Implication: even if the stage game has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which no stage's outcome is a Nash equilibrium

Infinitely Repeated Games: Example

Player 2

Player 1	L_2	R_2
L_1	1,1	5,0
R_1	0,5	4,4

- Trigger strategy
  – Play R_i in the first stage. In the t^{th} stage, if the outcome of all t-1 proceeding stages has been (R_1,R_2) then play R_i; otherwise, play L_i

Infinitely Repeated Games in Example (cont')

		Player 2	
		L_2	R_2
Player 1	L_1	1,1	5,0
	R_1	0,5	4,4

If any player deviates

$$V_d = 5+\delta\cdot 1+\delta^2\cdot 1+\cdots = 5+\frac{\delta}{1-\delta}$$

If no player deviates

$$V_c = \frac{4}{1-\delta} \quad (\text{Solve } V = 4+\delta V)$$

Condition for both players to play the trigger strategy (Nash equilibrium)

$$V_c \ge V_d \Rightarrow \frac{4}{1-\delta} \ge 5+\frac{\delta}{1-\delta} \Rightarrow \delta \ge \frac{1}{4}$$

Infinitely Repeated Games in Example (cont')

		Player 2	
		L_2	R_2
Player 1	L_1	1,1	5,0
	R_1	0,5	4,4

- The trigger strategy is a subgame perfect Nash equilibrium (Proof)
  – The infinitely repeated game can be grouped into two classes:
  – (1) Subgame in which all the outcomes of earlier stages have been (R_1,R_2)
    - Again the trigger strategy, which is Nash equilibrium of the whole game
  – (2) Subgames in which the outcome of at least one earlier stage differs from (R_1,R_2)
    - Repeat the stage-game equilibrium (L_1,L_2),which is also Nash equilibrium of the whole game

Friedman Theorem (1971)

- Also called Fork theorem
- Theorem: Let G be a finite, static game of complete information. Let $(e_1,\ldots,e_n)$ denote the payoffs from a Nash equilibrium of G, and $(x_1,\ldots,x_n)$ denote any other feasible payoffs from G. If $x_i>e_i$ for every player i and if δ is sufficiently close one, then there exists a subgame-perfect Nash equilibrium of the infinite repeated game $G(\infty,\delta)$ that achieve $(x_1,\ldots,x_n)$ as the average payoff

Friedman Theorem (cont')

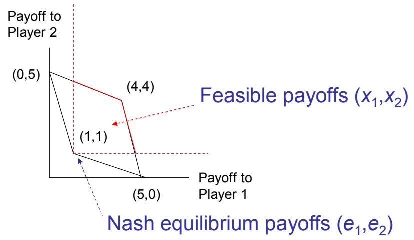
- Feasible playoff
  – the payoff is feasible in the stage game G if they are a convex combination (non –negative weighted average) of the pure-strategy payoffs of G
- Example
  – By playing (L_2,R_1) or (R_1,L_2) depending on a flip of a (fair) coin , they achieve the expected payoff (2.5,2.5)
  – Payoff (2.5,2.5) is feasible

		Player 2	
		L_2	R_2
Player 1	L_1	1,1	5,0
	R_1	0,5	4,4

Friedman Theorem (cont')

- Notation
  – Let $(a_{e1},\ldots,a_{en})$ be the Nash equilibrium of G that yields the equilibrium playoffs $(e_1,\ldots,e_n)$. Likewise, let $(a_{x1},\ldots,a_{xn})$ be the collection of actions that yields the feasible payoffs $(x_1,\ldots,x_n)$
- Trigger strategy
  – For player i, play a_{xi} in the first stage. In the tth stage, if outcome of all t-1 proceeding stages has been $(a_{x1},\ldots,a_{xn})$ then play a_{xi}; otherwise, play a_{ei}

Friedman Theorem (cont')



Friedman Theorem (cont')

Sustaining of a trigger strategy

If any player deviates

$$V_i^d = d_i + \delta \cdot e_i + \delta^2 \cdot e_i + \cdots = d_i + \frac{\delta}{1-\delta} e_i$$

If no player deviates

$$V_i^c = \frac{x_i}{1-\delta} \quad \text{(Solve } V_i^c = x_i + \delta V_i \text{)}$$

$$V_i^c \geq V_i^d \Rightarrow \frac{x_i}{1-\delta} \geq d_i + \frac{\delta}{1-\delta} \cdot e_i \Rightarrow \delta \geq \frac{d_i - x_i}{d_i - e_i}$$

Condition for all players to play the trigger strategy

$$\Rightarrow \quad \delta \geq \max_i \frac{d_i - x_i}{d_i - e_i}$$

Example 1: Collusion between Cournot Duopolists

- Trigger strategy
  - Produce half monopoly quantity, $q_m/2$, in the first period. In the tth period, produce $q_m/2$ if both firms have produced $q_m/2$ in each of the t-1 previous periods; otherwise, produce the Cournot quantity

Example 1: Collusion between Cournot Duopolists (cont')

Collusion profit $\frac{\pi_m}{2} = \frac{(a-c)^2}{8}$ Competition profit $\pi_C = \frac{(a-c)^2}{9}$

Deviation profit $\pi_d = \frac{9(a-c)^2}{64}$

$$\pi_d = \max_{q_d} \left(a - q_d - \frac{1}{2} q_m - c \right) q_d$$

Solve FOC $\Rightarrow q_d = \frac{3(a-c)}{8}$

$\Rightarrow Q = \frac{5(a-c)}{8}$

$\Rightarrow \pi_d = \frac{3(a-c)}{8} \cdot \frac{3(a-c)}{8}$

Condition for both producer to play trigger strategy

$$\frac{1}{1-\delta} \cdot \frac{1}{2} \pi_m \geq \pi_d + \frac{\delta}{1-\delta} \cdot \pi_C \quad \Rightarrow \quad \delta \geq \frac{9}{17}$$

Example 2: Efficiency Wages

- The firms induce workers to work hard by paying high wages and threatening to fire workers caught shirking (Shapiro and Stiglitz 1984)
- Stage game
  - First, the firms offers the worker a wage w
  - Second, the worker accepts or rejects the firm's offer
  - If the worker rejects w, then the worker becomes self-employed at wage w_0
  - If the worker accepts w, then the worker chooses either to supply effort (which entails disutility e) or to shirk (which entails no disutility)

Example 2: Efficiency Wages (cont')

- The worker's effort decision is not observed by the firm, but the worker's output is observed by both the firm and the worker
- Output can be either high (y) or low (0)
  - If the worker supplies effort then output is sure to be high
  - If the worker shirks then output is high with probability p and low with probability $1-p$
  - Low output is an incontrovertible sigh of shirking
- Payoffs: Suppose the firm employs the worker at wage w
  - if the worker supplies effort and output is high, the playoff of the firm is $y-w$ and playoff of the worker is $w-e$
- Efficient employment
  - $y-e>w_0>py$

Example 2: Efficiency Wages (cont')

- Subgame-perfect outcome
  - The firm offer $w=0$ and the worker chooses self-employment
  - The firms pays in advance, the worker has no incentive to supply effort
- Trigger strategy as repeated-game incentives
  - The firm's strategy: offer $w=w^*$ ($w^*>w_0$) in the first period, and in each subsequent period to offer $w=w^*$ provided that the history of play is high-wage, high-output, but to offer $w=0$, otherwise
  - The worker's strategy: accept the firm's offer if $w>w_0$ (choosing self-employment otherwise) the history of play, is high-wage, high-output (shirking otherwise)

Example 2: Efficiency Wages (cont')

- If it is optimal for the worker to supply effort, then the present value of the worker's payoff is

$$V_e = (w^* - e) + \delta V_e \Rightarrow V_e = (w^* - e)/(1-\delta)$$

- If it is optimal for the worker to shirk, then the (expected) present value of the worker's payoffs is

$$V_s = w^* + \delta\left\{pV_s + (1-p)\frac{w_0}{1-\delta}\right\} \Rightarrow V_s = \frac{(1-\delta)w^* + \delta(1-p)w_0}{(1-\delta p)(1-\delta)}$$

Example 2: Efficiency Wages (cont')

- It is optimal for the worker to supply effort if

$$V_e \geq V_s \Rightarrow w^* \geq w_0 + \frac{(1-p\delta)e}{\delta(1-p)} = w_0 + \left(1 + \frac{1-\delta}{\delta(1-p)}\right)e$$

- The firm's strategy is a best response to the worker's if

$$y > w^* \geq 0$$

- We assume $y - e > w_0$, the SPNE implies

$$y - e \geq w_0 + \frac{(1-\delta)e}{\delta(1-p)}$$

Example 3: Time-Consistent Monetary Policy

- Consider a sequential-move game in which employers and workers negotiate nominal wages, after which the monetary authority chooses the money supply, which in turn determined the rate of inflation
- Employers and workers will try to anticipate inflation in setting the wage
- Actual inflation above the anticipated level of inflation will erode the wage , causing employers to expand employment
- The monetary authority therefore faces a trade-off between the costs of inflation and benefits of reduced unemployment

Example 3: Time-Consistent Monetary Policy (cont')

- The model (Barro and Gordon 1983)
- First, employers form an expectation of inflation, π^e.Second, the monetary authority observes this expectation and chooses actual inflation π .
- The payoff to employers is $-(\pi^e - \pi)^2$
  - Employers achieve maximum payoff when anticipate inflation correctly
- The payoff to the monetary authority

$U(\pi, y) = -c\pi^2 - (y - y^*)^2$

The monetary authority would like inflation to be zero but output (y) to be at its efficient level (y^*)

Example 3: Time-Consistent Monetary Policy (cont')

- Suppose that actual output is the following function of target output and surprise inflation:

$$y = by^* + d(\pi - \pi^e)^2,\ b<1,\ d>0$$

- The monetary authority's payoff can be rewritten as

$$W(\pi, \pi^e) = -c\pi^2 - [(b-1)y^* + d(\pi - \pi^e)]^2$$

$$\frac{\partial W}{\partial \pi} = 0 \Rightarrow \pi^*(\pi^e) = \frac{d}{c+d^2}\left[(1-b)y^* + d\pi^e\right]$$

Example 3: Time-Consistent Monetary Policy (cont')

- Employers anticipate that the monetary authority will choose $\pi^*(\pi^e)$, employers choose π^e to maximize $-[\pi^*(\pi^e) - \pi^e]^2$

$$\pi^*(\pi^e) = \pi^e \Rightarrow \pi^e = \frac{d(1-b)}{c}y^* = \pi_s$$

- The monetary authority's payoff can be rewritten as

$$W(\pi, \pi^e) = -c\pi^2 - [(b-1)y^*]^2$$

$$\text{if } \pi = 0 \Rightarrow W(\pi, \pi^e) \text{ is maximized}$$

Example 3: Time-Consistent Monetary Policy (cont')

- Consider the infinitely repeated game in which both players share the discount factor δ
- Derive conditions under which $\pi = \pi^e = 0$, in every period is a subgame-perfect Nash equilibrium
  - The employer hold the expectation $\pi^e=0$ provided that all prior actual inflations have been $\pi=0$. Otherwise, the employer hold the expectation $\pi^e = \pi_s$
  - The monetary authority set $\pi=0$ provided all prior expectations have been $\pi^e=0$. Otherwise, The monetary authority set $\pi = \pi^*(\pi^e)$

Example 3: Time-Consistent Monetary Policy (cont')

- The monetary authority's strategy is a best response to the employers' updating rule if

$$\frac{1}{1-\delta}W(0,0) \geq W\left(\pi^*(0),0\right) + \frac{\delta}{1-\delta}W\left(\pi_S,\pi_S\right)$$

$$\Rightarrow \delta \geq \frac{c}{\left(2c+d^2\right)}$$

$$\frac{\partial \delta}{\partial c} > 0; \frac{\partial \delta}{\partial d} < 0$$

Practice: Create your own ideas

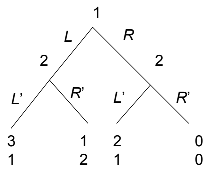
- Propose IT/IS applications, utilizing tournaments model
  - Competition rule
  - Uncertain factor
  - Effort and outcome functions
- Propose IT/IS applications, utilizing reliability/free-riding model
  - Commons (public goods) function
  - Individual (Nash) and Efficient (Social optimum) results
  - Incentive mechanism (fine, liability)
- IT/IS applications include EC,KM,SCM,…etc.
- Grading: will be considered to improve your participation parts
- When to submit : two weeks from today

Extensive-Form Representation of Games

- Definition: the extensive-form representation of a game specifies:
  - (1) the players in the game
  - (2a) when each player has the move
  - (2b) what each player can do at each of his or her opportunities to move
  - (2c) what each player knows at each of his or her opportunities to move, and
  - (3) the playoff received by each player for each combination of moves that could be chosen by the players

Extensive-Form Representation of Games (cont')

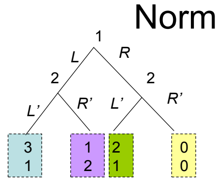
- Player 1 chooses an action a_1 from the feasible set $A_1=\{L,R\}$
- Players 2 observes a_1 and then chooses an action a_2 from the set $A_2=\{L',R'\}$
- Playoffs are $u_1(a_1,a_2)$ and $u_2(a_1,a_2)$, as shown in the game tree



Extensive-Form Representation of Games (cont')

- Definition: A strategy for a player is a complete plan of actions – it specifies a feasible action for the player in every contingency in which the player might be called on to act
- Action vs. strategy
  - Player 2 has 2 actions $A_2=\{L,R\}$ but 4 strategies $S_2=\{(L',L'), (L',R'), (R',L'), (R',R')\}$
    - (L',L'): if player 1 play L (R), then play L' (L')
    - (L',R'): if player 1 play L (R), then play L' (R')
    - (R',L'): if player 1 play L (R), then play R' (L')
    - (R',R'): if player 1 play L (R), then play R' (R')
  - Player 1 has 2 actions $A_1=\{L,R\}$ also has 2 strategies $S_1=\{L,R\}$

Transform Extensive-Form to Normal-Form



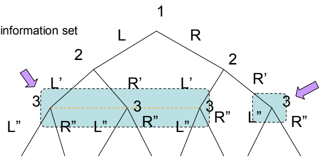
Player 2

		(L',L')	(L',R')	(R',L')	(R',R')
Player 1	L	3,1	3,1	1,2	1,2
	R	2,1	0,0	2,1	0,0

Information Set

- Definition: An information set for a player is a collection of decision nodes satisfying
  - (*i*) the player has the move at every node in the information set, and
  - (*ii*) when the play of the game reaches a node in the information set, and the player with the move does not know which node in the information ser has (or has not) been reached

Information Set: Example



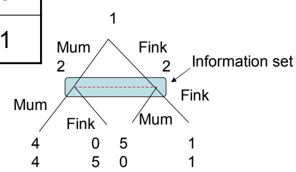
1. Player 1 choose an action a_1 from the feasible set A_1={*L*,*R*}
2. Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2={*L'*,*R'*}
3. Player 3 observes whether or not (a_1,a_2)=(*R*,*R'*) and then chooses an action a_3 from feasible set A_3={*L"*,*R"*}

⇨ Two information sets

Transform Normal Form to Extensive Form

Prisoner 2

		Mum	Fink
Prisoner 1	Mum	4,4	0,5
	Fink	5,0	1,1

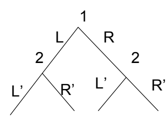


Interpretation of Prisoner 2's information set : when Prisoner 2 gets the move, all he know is that the information set has been reached (that prisoner 1 has move), not which node been reached (what prisoner 1 did)

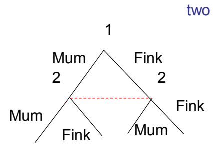
Subgame: Definition

- Definition : A subgame is an extensive-form game
  - (a) begins at a decision node *n* that is a singleton information set (but is not the game's first decision node)
  - (b) includes all the decision and terminal nodes following *n* in the game tree (but no nodes that do not follows), and
  - (c) does not cut any information sets (i.e. if a decision node *n'* follows *n* in the game tree, then all other nodes in the information set containing *n'* must also follow *n* , and so must be included in the subgame)

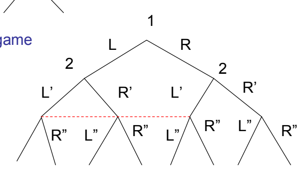
Subgame : Example



two subgame



No subgame

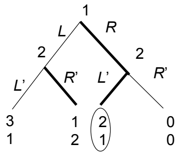


one subgame

Subgame-Perfect Nash Equilibrium vs. Outcome

- Definition: (Selten 1965) A Nash equilibrium is subgame-perfect if the players' strategies constitute a Nash equilibrium in every subgame
  - Equilibrium is a collection of strategies (and strategy is a complete plan of action), whereas an outcome describes what will happen only in the contingency that are expected to arise, not in every contingency that might arise
  - In the two-stage game of complete and perfect information, the backwards-induction outcome is $(a_1^*, R_2(a_1^*))$ but the subgame-perfect Nash equilibrium is $(a_1^*, R_2(a_1))$. $R_2(a_1^*)$ is an action and $R_2(a_1)$ is a strategy (best response function) for player 2

Subgame Perfect Nash Equilibrium vs. Nash Equilibrium



Backwards-induction outcome: (*R*,*L*')
Subgame perfect Nash equilibrium (*R*,(*R*',*L*'))

Nash equilibrium (*L*,(*R*',*R*'),*R*,(*R*',*L*'))

		Player 2			
		(*L*',*L*')	(*L*',*R*')	(*R*',*L*')	(*R*',*R*')
Player 1	*L*	3,1	3,1	1,2	1,2
	R	2,1	0,0	2,1	0,0

Subgame perfect Nash equilibrium v. Nash equilibrium (cont')

- Subgame–perfection eliminates Nash equilibria that reply on noncredible threats or promise
  - (*L*, (*R*',*R*')) : player 2's strategy is to play *R*' not only if player 1 chooses *L* but also if player 1 chooses *R*
  - The Nash equilibrium (*L*,(*R*',*R*')) is not a subgame-perfect since player 2's choice of *R*' (with playoff 0) is not optimal in the subgame beginning at player 2's decision node following *R* by player 1 (optimal choice is *L*' with playoff 1)

Homework #2

- Problem set
  - 3,6,8,14,17,20 (from Gibbons)
- Due date
  - two weeks from current class meeting
- Bonus credit
  - Propose new applications in the context of IT/IS or potential extensions from examples discussed